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LINK HOMOTOPY WITH MANY COMPONENTS

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INTRODUCTION

GIVEN nonnegative integers m and $p_1 = p; p_2, \ldots, p_r$, it is an outstanding problem to classify the link maps

$$f = f_1 \amalg f_2 \amalg \ldots \amalg f_r \colon S^{p_1} \amalg \ldots \amalg S^{p_r} \longrightarrow S^m$$

(i.e. the component maps $f_1, \ldots, f_r$ are continuous and have pairwise disjoint images) up to link homotopy (ie. up to deformations through such link maps).

This rather crude equivalence relation was introduced in 1954 by J. Milnor [12] in the dimension setting $m = 3$, $p_1 = p_2 = \ldots = p_r = 1$ in an attempt to get a first rough understanding of the overwhelming multitude of classical links. Milnor gave already a precise triviality criterion for such links in terms of his μ-invariants; only recently, though, the full classification in this setting has been given by N. Habegger and X. Lin.

In the last few years higher dimensional link homotopy has also attracted much new interest. One line of the development, pursued by W. Massey, D. Rolfsen, R. Fenn, P. Kirk, the author and U. Kaiser a.o., concentrated on link maps with two components; e.g. in a large metastable dimension range there is now an exact sequence at our disposal which reduces our classification problem to standard homotopy questions (see [11]). A step in a different direction was taken in 1984 which led to a whole hierarchy of generalized μ-invariants for higher dimensional link maps with an arbitrary number of components (see [7]). Now it turns out that $\mu = \mu(1, 2, 3)$, i.e. the first of these invariants which is truly of higher order, often fills the remaining gap.

MAIN THEOREM. *Assume* (∗) $q := \{\max p_2, \ldots, p_r\} \le \frac{2}{3}m - 1$ *and* (∗∗) $p_1 \le 3(m - q - 2)$ *and exclude the case* $m = 3 \le p_1 + p_2 + \ldots + p_r$. *Then we have the isomorphism of abelian groups*

$$BLM^m_{p_1, \ldots, p_r} \xleftrightarrow{\;\cong\;} \bigoplus_{1 \le i < j \le r} BLM^m_{p_i, p_j} \oplus \bigoplus_{1 \le i < j < k \le r} \pi^S_{p_i + p_j + p_k - 2m + 3}$$

which takes a link homotopy class $[f] = [f_1 \amalg f_2 \ldots \amalg f_r]$ *to* $\oplus [f_i \amalg f_j] \oplus \bigoplus \mu[f_i \amalg f_j \amalg f_k]$.

Here it is convenient to eliminate complications which arise when one component of a link map is already linked with just the base point of another one. The resulting set $BLM^m_{p_1, \ldots, p_r}$ of "base point preserving" link homotopy classes coincides often with its base point free counterpart (see Proposition 1.7). In particular, all the sets $BLM^m_{p_i, p_j}$ on the right hand side of the bijection above agree with the corresponding sets $LM^m_{p_i, p_j}$ which, in turn, have been determined in [11]. Thus our theorem reduces the calculation of $BLM^m_{p_1, \ldots, p_r}$ to problems in standard homotopy theory.

Basepoint preserving link homotopy provides also the natural setting for connected sum operations which we can define frequently even if some of the dimensions p_i are zero; e.g. in the extreme case $p_2 = \dots = p_r = 0$ we have the canonical isomorphism

$$e_* : \pi_p\left(\bigvee_2^r S^{m-1}\right) \longrightarrow BLM^m_{p,0,\dots,0}$$

whenever $m \geq 3$, and our theorem just yields the Hilton decomposition. The reader should be warned, however, that sometimes there exists no reasonable sum operation (e.g. on $BLM^2_{1,0,\dots,0}$ for $r \geq 3$, see example 1.6) or that it may be noncommutative (as detected by μ; e.g. on $BLM^3_{p_1,\dots,p_r}$ when all $0 \leq p_i \leq 1$ and $\sum_{i=1}^r p_i \geq 3$). Also, additive inverses are hard to find in link homotopy (as opposed to *link concordance*); when we can show that $BLM^m_{p_1,\dots,p_r}$ is a full group, then often only after having gone half the way towards computing it. Nevertheless, abelian group structures (if they exist) are of course very valuable, e.g. for extending the domain of μ which is originally defined only on certain ("κ_2-trivial") link maps.

As a striking consequence of our theorem note that link maps are entirely determined, up to link homotopy, by their sub-link maps with three components. This is all the more surprising since link homotopy has no lifting principle: in general a link homotopy of a sub-link map $f_1 \amalg \dots \amalg f_s$, $s < r$, does not extend to a full link homotopy of $f_1 \amalg \dots \amalg f_r$ (consider e.g. the case $s = 1$, $r = 2$); nor can homotopies of sub-links be fitted together to yield a full link homotopy.

In any case, a direct homotopy theoretical approach to link homotopy seems to hold little promise, and so the proof of the main theorem gets us deeply involved in differential topology. The first step centers around the $\tilde{\beta}$-invariants: they are obstructions (and, as we show in §2, often the only ones) to making $f_2 \amalg \dots \amalg f_r$ into a smooth embedding. It is here that we need the (highly unsymmetrical) dimension requirements of the theorem decisively for numerous embedding, isotopy and disjointness arguments. They imply as a side result that the $\tilde{\beta}$-invariants of a full link map depend only on its 2-component sub-link maps.

As a second step, we classify link maps which embed $S^{p_2} \amalg \dots \amalg S^{p_r}$. This is based strongly on the results of [11] and on an analysis of our μ-invariant (see §3).

Example. Given $q \geq 4$, consider the homomorphism

$$(\tilde{\beta}_{1,2}; \alpha_{1,2}, \alpha_{1,3}, \alpha_{2,3}; \mu) : BLM^{2q}_{2q-2,q,q-1} \longrightarrow \bigoplus_{k=1}^{\infty} \mathbb{Z} \oplus \pi^S_{q-1} \oplus \pi^S_{q-2} \oplus \pi^S_0 \oplus \mathbb{Z}$$

where $\tilde{\beta}_{1,2}\{f_1 \amalg f_2 \amalg f_3\} \underset{\text{def}}{=} \tilde{\beta}[f_1 \amalg f_2]$ coincides with P. Kirk's σ-invariant (see [6]) and similarly $\alpha_{i,j}$ is defined by the "generalized (first order) linking number" α (see e.g. [17], [14], [8] or also 2.4). Then both $\tilde{\beta}_{1,2} \oplus \alpha_{1,2}$ and the full homomorphism above have cokernel $\mathbb{Z}/(1 + (-1)^q)\mathbb{Z}$ and a kernel which is $\mathbb{Z}_2$ for odd $q \neq 7$ and trivial otherwise (by Adams' work on the Hopf invariant).

Remark. In the special case of *embedded* links with three components of codimension at least three W. Massey has also obtained a result which is closely related to our main theorem (see [13]).

§1. BASE POINT PRESERVING LINK MAPS

Throughout this paper we fix integers $m \geq 2$ and $p_1, \ldots, p_r \geq 0$ where $r \geq 1$ is arbitrary. We use the following abbreviations consistently:

$$(p) = (p_1, p_2, \ldots, p_r)$$

$$p = p_1$$

and

$$q = \max\{p_2, \ldots, p_r\}.$$

Unit spheres are equipped with the base point $* = (-1, 0, \ldots, 0)$. We will also use the points

$$*_i = (i, 0, \ldots, 0), \quad i = 1, 2, \ldots, r,$$

in the halfspace $\mathbb{R}^m_+$ which is defined by the inequality $x_2 \geq 0$; as usual, $x_1, x_2, \ldots$ denote the coordinates of Euclidean space.

Definition 1.1. *A base point preserving link map* is a continuous map

$$f = f_1 \amalg f_2 \amalg \ldots \amalg f_r : S^{p_1} \amalg \ldots \amalg S^{p_r} \longrightarrow \mathbb{R}^m_+$$

such that $f_i(*) = *_i$ for $i = 1, 2, \ldots, r$ and the component maps have pairwise disjoint images, $f_i(S^{p_i}) \cap f_j(S^{p_j}) = \phi$ for $1 \leq i < j \leq r$.

A *base point preserving link homotopy* is a continuous map

$$F = F_1 \amalg F_2 \amalg \ldots \amalg F_r : (S^{p_1} \amalg \ldots \amalg S^{p_r}) \times I \longrightarrow \mathbb{R}^m_+$$

which restricts to a base point preserving link map at every level t of the unit interval I.

The resulting *set of all base point preserving link homotopy classes* is denoted by $BLM^m_{(p)}$.

For every subsequence (p') of (p) the two maps

$$BLM^m_{(p')} \underset{\text{rest}}{\overset{\text{incl}}{\rightleftarrows}} BLM^m_{(p)} \tag{1.2}$$

are canonically defined by forgetting component maps or adding constant ones (and shrinking or stretching the x_1-axis appropriately to obtain the correct base point behaviour). Clearly, rest $\circ$ incl is the identity on $BLM^m_{(p')}$.

Example 1.3. If $q \leq m - 2$, the following simple construction produces useful elements of $BLM^m_{(p)}$. Consider the embeddings

$$e_i : (S^{p_i}, *) \hookrightarrow (\mathbb{R}^m_+, *_i), \quad i = 2, \ldots, r,$$

given by $e_i(x_1, x_2, \ldots, x_{p_i+1}) = (i, x_1 + 1, x_2, \ldots, x_{p_i+1}, 0, \ldots, 0)$. The boundary sphere of the obvious normal $(m - p_i)$-dimensional ε-ball at $e_i(1, 0, \ldots, 0)$ contains the point $(i, 2 + \varepsilon, 0, \ldots, 0)$ which we move linearly first to $(1, 2 + \varepsilon, 0, \ldots, 0)$ and then to $*_1$. This procedure yields a map

$$e : V = \bigvee_{i=2}^{r} S^{m-p_i-1} \longrightarrow \mathbb{R}^m_+ - \coprod_{i=2}^{r} e_i(S^{p_i})$$

which takes the common base point $*$ to $*_1$. We define

$$e_* : \pi_p(V) \longrightarrow BLM^m_{(p)}$$

by $e_*[g] = [e \circ g \amalg e_2 \amalg \ldots \amalg e_r]$. Adding or collapsing various component spheres of the wedge V commutes, via e_*, with the corresponding maps incl and rest as defined in 1.2. ■

Next we turn to the additive structure.

PROPOSITION 1.4. *Let $m \geq 3$ and assume $p_1, p_2, \ldots, p_r \geq 1$ or $p_1, p_2, \ldots, p_r \leq m - 2$ or $q \leq m - 3$. Then there is a canonical sum operation which makes $BLM^m_{(p)}$ into a semigroup and which makes the maps incl, rest (see (1.2)) and also e_* (for $q \leq m - 2$, see 1.3) into semigroup homomorphisms. If $m - 2, p_1, p_2, \ldots, p_r \geq 2$ or $q \leq m - 3$, then $BLM^m_{(p)}$ is abelian.*

Proof. Let $T_{\substack{+ \\ (-)}}: \mathbb{R}^m_+ \longrightarrow \{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m_+ \mid {}_{(}\underset{-}{+}{}_{)} x_3 \geq 0\}$ be the two homeomorphisms which map the point $(x_1, \rho \sin\varphi, \rho\cos\varphi, x_4, \ldots, x_m)$ to $\left(x_1, \rho\sin\left(\frac{\varphi(+\pi)}{2}\right), \rho\cos\left(\frac{\varphi(+\pi)}{2}\right), x_4, \ldots, x_m\right)$ for $\rho \geq 0$ and $0 \leq \varphi \leq \pi$. Thus these maps shrink each (x_2, x_3)-halfplane in $\mathbb{R}^m_+$ onto its two quadrants. Clearly, T_+, T_- and the identity map are isotopic through homeomorphisms *into* $\mathbb{R}^m_+$.

If $p_1, \ldots, p_r \geq 1$, define the halfspheres $S^{p_i}_+$ and $S^{p_i}_-$ by the inequalities $x_2 \geq 0$ and $x_2 \leq 0$. Given two elements of $BLM^m_{(p)}$ we may find representatives f and g such that

$$f_i(S^{p_i}_-) = \{*_i\} = g_i(S^{p_i}_+) \quad \text{for } i = 1, 2, \ldots, r \tag{1.5}$$

and the remaining images intersect the boundary hyperplane $\partial\mathbb{R}^m_+$ only in $\{*_1, \ldots, *_r\}$. Define the link map $f + g$ by

$$(f+g)_i|S^{p_i}_+ = T_+ \circ f_i|S^{p_i}_+ \quad \text{and} \quad (f+g)_i|S^{p_i}_- = T_- \circ g_i|S^{p_i}_-$$

for $i = 1, \ldots, r$. The resulting class in $BLM^m_{(p)}$ is independent of the choices of representatives and is denoted by $[f] + [g]$.

If $p_1, \ldots, p_r \leq m - 2$, let (p') be the subsequence of (p) which omits all vanishing dimensions $p_i = 0$. Then by a transversality argument the maps incl and rest defined in 1.2 are bijective and inverse to one another; we equip $BLM^m_{(p)}$ with the sum operation induced from $BLM^m_{(p')}$.

The remaining case when $p \geq 1$ and $q \leq m - 3$ requires more care with the 0-dimensional components. Given any element in $BLM^m_{(p)}$, we may represent it by a link maps f which has the following three properties for every i such that $p_i = 0$:

(i) $f_i(1, 0, \ldots, 0) = (i, 2, 0, \ldots, 0)$;

(ii) for all $j \geq 2, j \neq i$, the image of f_j does not intersect some compact ε-neighborhood of the halfdisk

$$D_{i+} := \{(i, x_2, x_3, 0, \ldots, 0) \in \mathbb{R}^m_+ \mid x_2^2 + x_3^2 \leq 4\};$$

(iii) there is a compact ballbundle B_i smoothly embedded in $S^p - \{*\}$ such that $f_1(B_i)$ lies in the ε-sphere around the point $(i, 2, 0, \ldots, 0)$, $f_1(\partial B_i) = \{(i, 2 + \varepsilon, 0, \ldots, 0)\}$ and $f_1(S^p - B_i) \cap D_{i+} = \phi$. Indeed, a suitable isotopy of $\mathbb{R}^m_+$ brings f_i into the desired "standard form". Then we make the other component maps transverse to the line segment L_i which joins the two image points of f_i. Let B_i be given by a small tubular neighborhood of $f_1^{-1}(L_i)$ in S^p. Then the maps $f_1|(S^p - B_i)$ and f_j, $2 \leq j \neq i$, avoid L_i altogether and can even be pushed off D_{i+}.

Now given again two elements of $BLM^m_{(p)}$, pick representatives f and g which satisfy (i), (ii), (iii) and—for strictly positive dimensional components—the conditions required in 1.5.

We define the jth component map of $f+g$ as before whenever $j \geq 2$ and $p_j \geq 1$. However, $(f+g)_i$ is given the "standard form" as in condition (i) whenever $p_i = 0$. Thus, in order to avoid trouble with the extra "free point" in $S^0 \vee S^0$, we do *not* apply T_+ to f_i nor to $f_1|(B_i - C_i)$, but only to $f_1|(S^p - B_i)$; here C_i denotes a small collar of ∂B_i in B_i on which we define $(f+g)_1$ using the obvious rotation by 45° in the (x_2, x_3)-plane. We apply the same procedure to g and end up also with $(f+g)_1|S^p_-$. Although the jth component maps of f and g are moved entirely into different x_3-halfspaces while f_1 and g_1 are not, conditions (ii) and (iii) above guarantee that no undesirable intersections occur. Since $q \leq m-3$ these conditions are compatible with link homotopies (see also example 1.6 below), so that we obtain again a well defined sum operation on $BLM^m_{(p)}$.

In all three dimension cases the addition is associative and has a unit represented by the componentwise constant link map. As for commutativity assume $m \geq 4$ and let R denote the involution on $\mathbb{R}^m_+$ which rotates the (x_3, x_4)-plane by 180° and leaves all other coordinates fixed. Clearly R is isotopic to the identity and preserves the properties (i), (ii) and (iii); also $R \circ T_\pm \circ R = T_\mp$. Thus given representatives f and g as above, their sum is link homotopic to the link map $R(Rf + Rg)$ which yields $[g+f]$ as soon as condition 1.5 holds with opposite signs whenever $p_i \neq 0$. If $p_i \geq 2$, this can be achieved by a suitable rotation in the domain S^{p_i}. If $p \geq 2$, $q \leq m-3$ and $p_i = 1$, we may assume from the start, in analogy to the properties (i), (ii) and (iii) above, that $f_i = e_i$ is the standard embedding onto the unit circle around $(i, 1, 0, \ldots, 0)$ in D_{i+}, that $f_1(B_i)$ lies in the ε-sphere which is normal even to this circle at the point $(i, 2, 0, \ldots, 0)$ and that the remaining component maps f_j avoid even a 4ε-neighborhood of D_{i+} in $\mathbb{R}^m_+$, and similarly for g. Then the i-th component map of $R(Rf + Rg)$ has the same image as $(g+f)_i$, namely $S^1_+ \vee S^1_-$, but it runs through the two circles in the wrong order. Lifting $S^1_\pm$ to the level $x_4 = \pm 2\varepsilon$ and sliding each of these two circles (together with $f|B_i$ etc.) into the opposite x_3-halfplane leads to the necessary correction. To complete the proof of our proposition, we discuss the following case for arbitrary $m \geq 2$ and $p \geq 0$.

Example 1.6: $q = 0$. The map

$$e_*: \pi_p\left(\bigvee_{i=2}^{r} S^{m-1}\right) \longrightarrow BLM^m_{p,0,\ldots,0}$$

(see 1.3) is onto and for $(m, p) \neq (2, 1)$ or $r \leq 2$ even an isomorphism. Indeed, let $\tilde{C}_r$ denote the configuration space of ordered r-tuples of pairwise distinct points in $\mathring{\mathbb{R}}^m_+$. Recall from [1] that the projection $\tilde{C}_r \to \tilde{C}_{r-1}$ to the last $r-1$ components is a locally trivial fibration with fiber $\mathring{\mathbb{R}}^m_+ - \coprod_{i=2}^{r} e_i(S^0)$. Moreover, e is homotopic to a homeomorphism which identifies the wedge V with a deformation retract of $\mathbb{R}^m_+ - \coprod_{i=2}^{r} e_i(S^0)$. Now a link map f which maps only the r basepoints $*$ into $\partial\mathbb{R}^m_+$ determines (and is determined by) the map

$$\hat{f} = (f_1, f_2(-*), \ldots, f_r(-*)) : S^p - \{*\} \to \tilde{C}_r.$$

Any path in $\tilde{C}_{r-1}$ from $(f_2(-*), \ldots, f_r(-*))$ to the base point $(e_2(-*), \ldots, e_r(-*))$ lifts to a homotopy of $\hat{f}$ which shows $[f]$ to lie in the image of e_*. If $m \geq 3$, $\tilde{C}_{r-1}$ is even simply connected and a similar lifting argument applies to link homotopies so that e_* is also injective.

The case $m = 2$, $(p) = (1, 0, \ldots, 0)$, $r \geq 3$ is very different. Indeed, every smooth loop in the $(r-1)$-configuration space of the open upper halfplane $\mathring{\mathbb{R}}^2_+$ extends in a natural way to an isotopy of $\mathbb{R}^2_+$ which leaves the x_1-axis $\partial\mathbb{R}^2_+$ fixed. Therefore the pure braid group

$\pi_1(\tilde{C}_{r-1})$ acts on the homotopy groups of

$$V = \bigvee_{i=2}^{r} S^1 \sim \mathbb{R}^2_+ - \coprod_{i=2}^{r} e_i(S^0)$$

by automorphisms. Clearly, e_* induces a bijection from the resulting set of orbits in $\pi_p(V)$ onto $BLM^2_{p,0,0,\ldots,0}$. Now let a and b denote the first two standard generators in $\pi_1(V)$ circling around $e_2(-*)$ and $e_3(-*)$, resp., in the counterclockwise fashion. Then a rotation of these two center points around each other shows that $e_*(a) = e_*(b^{-1}ab)$. On the other hand, $e_*(c) \neq e_*(ab)$ for all $c \neq ab$, $c \in \pi_1(S^1_{(2)} \vee S^1_{(3)})$, since ab can be represented by a loop which goes around $e_2(-*)$ and $e_3(-*)$ simultaneously and which remains unaffected by rotations of these two points. It follows that $BLM^2_{1,0,0,\ldots,0}$ admits no sum operation which is compatible with e_*. ■

Finally, let us compare $BLM^m_{(p)}$ with the corresponding (base-point free) link homotopy set $LM^m_{(p)}$ of all link maps into S^m. After a suitable isotopy of S^m such a link map f takes the basepoint of S^{p_i} to the prescribed value

$$*_i \in \mathbb{R}^m_+ \subset \mathbb{R}^m \subset S^m = \mathbb{R}^m \cup \{\infty\}$$

for all $1 \leq i \leq r$ (and also avoids the point ∞ if $r \geq 2$); however, the linking behaviour of one component map with just the remaining basepoints may prevent us from deforming f entirely into the halfspace $\mathbb{R}^m_+$. Thus the natural forgetful map

$$\text{forg} : BLM^m_{(p)} \longrightarrow LM^m_{(p)}$$

is onto if $p_1, \ldots, p_r \leq m-2$ or if $r = 2$ (compare [9], §1), but not in general; e.g. if $r \geq 3$ and $\pi_{p_i}(S^{m-1}) \neq 0$ for some $1 \leq i \leq r$, then f_i can separate any two remaining component maps nontrivially.

PROPOSITION 1.7. *forg is bijective in the following two cases:*

(i) $p_1, \ldots, p_r \leq m-3$; *or*
(ii) $r = 2$ *and* (p_1 *or* $p_2 \leq m-3$ *or* $p_1 = p_2 = m-2$).

However, forg cannot be injective if the addition in $BLM^m_{(p)}$ *defined by proposition* 1.4 *is noncommutative (this holds e.g. if there are distinct indices* $1 \leq i, j, k \leq r$ *such that* $p_i = p_j = m-2 \geq p_k = 1$).

For the proof see Corollary 3.12 and [8], 2.3. If forg is bijective, then the sum operation in $BLM^m_{(p)}$ defined by Proposition 1.4 gives rise to a canonical addition in $LM^m_{(p)}$ which must be commutative. In the case $r = 2$ it generalizes the addition used e.g. in [17] and in [8].

§2. THE $\tilde{\beta}$-INVARIANTS AND EMBEDDING CRITERIA

In this section we describe a method for extending the basic link homotopy invariants from the case $r = 2$ to link maps with any number of components. We encounter embeddability obstructions which play a central rôle in our link homotopy classification.

Given a link map

$$f = f_1 \amalg f_2 \amalg \ldots \amalg f_r : S^{p_1} \amalg \ldots \amalg S^{p_r} \longrightarrow \mathbb{R}^m$$

with arbitrary base point behaviour, we construct first a map

$$v_f : (S^m, \infty) \longrightarrow \left(\bigvee_{i=1}^{r} \text{Map}(S^{p_i}, S^m), * \right) \tag{2.1}$$

whose homotopy class depends only on the link homotopy class of f. (One can show that the weaker invariant

$$[v_f] \in \pi_m\left(\bigvee_{i=1}^{r} QS^{m-p_i}\right), \tag{2.2}$$

is determined by, and determines, the framed link bordism class of f, see definition 1.11 in [10]. Here we equip the space $\mathrm{Map}(S^{p_i}, S^m)$ of all continuous maps from S^{p_i} to S^m with the compact-open topology and with the constant map at $\infty \in S^m = \mathbb{R}^m \cup \{\infty\}$ as the basepoint, and we use the inclusion into the iterated loop space $QS^{m-p_i} := \Omega^N S^{N+m-p_i}$, $N \gg 0$). Choose a number $\varepsilon > 0$ which is smaller than half the distance between any two image points of different components of f, and a map $d : (B_\varepsilon, \partial B_\varepsilon) \to (S^m, \infty)$ which restricts to an orientation preserving diffeomorphism from the open ε-ball $B_\varepsilon - \partial B_\varepsilon$ around 0 onto $\mathbb{R}^m$. For $1 \le i \le r$ we define v_f on the ε-neighborhood of $f_i(S^{p_i})$ in $\mathbb{R}^m$ by

$$v_f(x)(y) = \begin{cases} d(x - f_i(y)) & \text{if } \|x - f_i(y)\| \le \varepsilon, \\ \infty & \text{else} \end{cases}$$

where $x \in \mathbb{R}^m$ and $y \in S^{p_i}$; outside of these disjoint ε-neighborhoods v_f takes the constant value ∞.

Now fix $1 \le j \le r$ and collapse the j^{th} component in the wedge in 2.1 to obtain the map

$$v_{f,j} : \mathbb{R}^m \longrightarrow \bigvee_{\substack{1 \le i \le r \\ i \ne j}} \mathrm{Map}(S^{p_i}, S^m) \tag{2.3}$$

which is trivial on a neighborhood U of $f_j(S^{p_j})$. Thus, given any map $h : N \to U$, we can shrink it to $f_j(*)$ and obtain a map from the suspension SN^+ to the wedge above. Since there are several canonical ways to produce such a map into U (see [10], 1.1 and 3.2), we get interesting link homotopy invariants. E.g. if we take $h = f_j$, we obtain the link homotopy invariant

$$\alpha_j(f) \in \pi_{p_j+1}\left(\bigvee_{\substack{1 \le i \le r \\ i \ne j}} \mathrm{Map}(S^{p_i}, S^m)\right); \tag{2.4}$$

in the special case $r = 2$ both $\alpha_1(f)$ and $\alpha_2(f)$, together with the inclusions into infinite loop spaces, yield the invariant

$$\pm\, \alpha(f) \in \pi_{p_j+1}(QS^{m-p_i}) = \pi^S_{p_1+p_2+1-m}$$

which was discussed e.g. in [17] (where α is denoted by S), [14] and [8].

In this paper the choice of h as constructed in §1 of [11] will play a central rôle, and we adopt the notation used there. Thus if $p_j \le m - 2$, we may assume after a link homotopy that f_j is a smooth selftransverse immersion regularly homotopic to the standard embedding $S^{p_j} \subset \mathbb{R}^m$ (see [3]). Let $\tilde{D}$ and D be the double point manifolds which immerse onto the double point loci of f_j in S^{p_j}, and $\mathbb{R}^m$. If λ_D denotes the line bundle over D which is associated to the double cover $\tilde{D}$, we take $N = \hat{\lambda}_D$ to be its Thom space and define $h : \hat{\lambda}_D \to \mathbb{R}^m$ by a nulhomotopy of $\tilde{D}$ in S^{p_j}, composed with f_j. Thus a careful analysis of the double points of f_j, together with (2.3), yields the following data:

(i) the zerobordant $(m - p_j)$ λ-manifold D of dimension $2p_j - m$ (see [11], 1.8); and
(ii) the basepoint preserving map

$$g : S\hat{\lambda}_D \,(= \text{Thomspace of } \lambda_D \oplus \mathbb{R}) \longrightarrow \bigvee_{\substack{1 \le i \le r \\ i \ne j}} QS^{m-p_i} \tag{2.5}$$

which is induced from $v_{f,j}$ and a basepoint preserving nulhomotopy H of h.

We define

$$\tilde{\beta}_j(f) := [D, g] \in B_j(m, (p)). \tag{2.6}$$

Here $B_j(m, (p))$ denotes the joint bordism group of all pairs (D, g) as in (i) and (ii), or equivalently, of all diagrams

$$\begin{tikzcd}
C_1 \arrow[rrd, "b_1"] & & \\
\vdots & & \lambda_D \oplus \mathbb{R} \arrow[dd] \\
C_i \arrow[rru, "b_i"] & & \\
\vdots & & \\
C_r \arrow[rruuu, "b_r"] & & D
\end{tikzcd} \quad , \quad i \neq j, \tag{2.7}$$

where D is as in (i), each C_i is a closed $(m - p_j - 1)\lambda$-manifold of dimension

$$s_{i,j} = p_i + 2p_j + 2 - 2m$$

and the maps b_i, $i \neq j$, have pairwise disjoint images and pull the structure line bundle λ_D back to λ_{C_i} (compare [11], 1.5). Indeed, g, when restricted to $U_i = g^{-1}(QS^{m-p_i} - \{*\})$, corresponds after a small approximation to a smooth map $\bar{g} : U_i \times S^{p_i+N} \to S^{m+N}$ having $0 \in \mathbb{R}^m$ as a regular value; we put $C_i = \bar{g}^{-1}\{0\}$ and define b_i by the projection into $U_i \subset \lambda_D \oplus \mathbb{R}$. This correspondence (compare [16]) extends the classical Thom-Pontrjagin isomorphism which assigns disjoint *embeddings* b_i to a map $g : S\hat{\lambda}_D \to \vee S^{m-p_i}$. In the case of $\tilde{\beta}_j(f)$, the manifold C_i is given by the intersection of H with f_i; so our invariant measures to some extent how the map h (which is associated to the double points of f_j) links with the other component maps of f.

PROPOSITION 2.8. *Fix $1 \leq j \leq r$ and assume $p_j \leq m - 2$.*

Then $\tilde{\beta}_j(f)$ is a welldefined invariant depending only on the (basepoint free) link homotopy class of f in $S^m = \mathbb{R}^m \cup \{\infty\}$.

If f_j is an embedding, then $\tilde{\beta}_j(f) = 0$.

If f_i is an embedding for some $i \neq j$, then $\tilde{\beta}_j(f)$ has a representative as in 2.7 with b_i being an embedding.

The analogous statement holds for the refined version of $\tilde{\beta}_j(f)$ where g maps into $\bigvee_{i \neq j} \operatorname{Map}(S^{p_i}, S^m)$ (compare 2.5 and 2.3). The proof follows the lines of [11], 1.11. ■

The target group $B_j(m, (p))$ of $\tilde{\beta}_j$ can be expressed also as a (reduced) stable homotopy group of a Thom space of a vector bundle whose base itself fibers over P^∞ with fiber $\Omega^2\left(\bigvee_{i \neq j} QS^{m-p_i}\right)$. The details of this description have been given for $r = 2$ in [11], 1.2. Fortunately, often this is all we need even in the case of an arbitrary number of components. Indeed, for $i \neq j$ consider the homomorphisms

$$B_j(m, (p)) \rightleftarrows B(m, p_i, p_j) \xrightarrow{c_i} \Omega_{s_{i,j}}(P^\infty; (m - p_j - 1)\lambda) \tag{2.9}$$

defined by the obvious inclusion and collapsing maps in the wedge of loop spaces in 2.5, and by just taking the $(m - p_j - 1)\lambda$-manifold C_i in 2.7. Here $\Omega_*(P^\infty; (m - p_j - 1)\lambda)$ denotes the normal bordism of all such manifolds or, equivalently, the stable homotopy of the stunted projective space P^∞/P^{m-p_j-2} (compare §1 in [11]; the (stabilized) joint bordism group $B(m, p_i, p_j)$ agrees with $B_2(m, p_i, p_j)$.)

PROPOSITION 2.10. *Fix $2 \le j \le r$ and assume $p_i + 2p_j + p_k \le 3m - 5$ for all $2 \le i \le r$ and $1 \le k \le r$ such that $i \neq j \neq k$. Then the homomorphisms in 2.9 determine isomorphisms*

$$B_j(m,(p)) \underset{\text{rest}_*}{\overset{\text{incl}_*}{\rightleftarrows}} B(m,p_1,p_j) \oplus \bigoplus_{\substack{i=2\\ i\neq j}}^{r} \Omega_{\cdot_{i,j}}(P^{\infty};(m-p_j-1)\lambda)$$

which are inverse to one another.

Proof. Since c_i is bijective for $1 < i \neq j$ (see [11], 1.14), we need to show only that rest_* has a trivial kernel. Given an element of this kernel, we may represent it by a diagram as in 2.7 such that $b_2, \ldots, b_r$ are disjoint embeddings into the zero section D of $\lambda_D \oplus \mathbb{R}$ which do not meet the projection of $b_1(C_1)$, either; this follows in our dimension setting from standard embedding, isotopy and transversality arguments. Moreover, we can identify a tubular neighborhood of C_i in D with the cokernel bundle of a monomorphism from λ_{C_i} into $C_i \times \mathbb{R}^{m-p_i-1}$. But this can be extended over a zero bordism of C_i and glued to D. We obtain a representative with empty $C_2, \ldots, C_r$; hence it comes from an element in $B(m,p_1,p_j)$ which in turn is trivial by assumption. ■

Clearly, $\tilde{\beta}_j$ makes the maps incl and rest (cf. 1.2) and their base point free counterparts commute with the corresponding inclusion and collapsing homomorphisms as in 2.9. E.g. in the language of the last proposition

$$\text{rest}_*(\tilde{\beta}_j(f)) = \left(\tilde{\beta}(f_1 \amalg f_j), \bigoplus_{\substack{i=2\\ i\neq j}}^{r} \beta(f_i \amalg f_j)\right) \tag{2.11}$$

whenever $p_j \le m-2$; here $\beta := c \circ \tilde{\beta}$ denotes a very crude version of $\tilde{\beta} := \tilde{\beta}_2$ which depends even only on the α-invariant via a Hopf invariant homomorphism (see [8], §4).

Example 2.12. Let $m = p_1 + 2 = 2p_2 \ge 4$. We have

$$\tilde{\beta}(f_1 \amalg f_2) \in \bigoplus_{1}^{\infty} \mathbb{Z}\ ; \quad \beta(f_1 \amalg f_2) \in \mathbb{Z}/(1+(-1)^{p_2})\mathbb{Z}\ ; \quad \alpha(f_1 \amalg f_2) \in \pi^S_{p_2-1};$$

and $\beta \neq 0$ precisely for $p_2 = 2, 4$ or 8; on the other hand, $\tilde{\beta}$ is identical with Kirk's σ-invariant here whose values form an infinitely generated group (see [5], §3, and [6], 4.3).

THEOREM 2.13. *Assume*

$$(*)\quad q \le \frac{2}{3}m - 1 \quad \textit{and} \quad (**)'\ p \le 3(m-q-2)+1.$$

Then the following three statements are equivalent for every link homotopy class $[f] \in (B)LM^m_{(p)}$ both in the base point free and in the base point preserving setting:

(i) *$[f]$ has a representative which embeds $S^{p_2} \amalg \ldots \amalg S^{p_r}$;*
(ii) *$\tilde{\beta}_j(f) = 0$ for $j = 2, \ldots, r$;*
(iii) *$\tilde{\beta}(f_1 \amalg f_j) = 0$ for all $2 \le j \le r$ and*
$\beta(f_i \amalg f_j) = 0$ for all $2 \le i, j \le r$ such that $i \neq j$.

Proof. In view of 2.8 and 2.11 it suffices to establish the following more precise result: if f is a link map such that $f_2, \ldots, f_r$ are selftransverse immersions and $\tilde{\beta}_j(f) = 0$ for an arbitrary fixed j, $2 \le j \le r$, then there exists a link homotopy of f which deforms f_j into an embedding and which leaves $f_2, \ldots, f_{j-1}, f_{j+1}, \ldots, f_r$ unchanged. This is proved as in the case of two components (see the proofs of Proposition 1.18 and of Theorem 1.15 in [11] applied to $f_i \amalg f_j$ when $i \geq 2$, $i \neq j$, and when $i = 1$, resp.). Our dimension assumptions are needed already for such 2-component sublinks. Fortunately, they also allow us to deal disjointly with the different component maps f_i. This follows from transversality and isotopy arguments in the spirit of the previous proof.

Note that our link homotopy may be chosen to avoid the point ∞ in $S^m = \mathbb{R}^m \cup \{\infty\}$ if f does. ■

Next a standard argument relates the embedding question to the map e_* defined in 1.3.

PROPOSITION 2.14. *Assume $q \leq m - 3$. Then a link homotopy class $u \in BLM^m_{(p)}$ lies in $e_*\left(\pi_p\left(\bigvee_{i=2}^{r} S^{m-p_i-1}\right)\right)$ if and only if u has a representative f such that $f_2 \amalg f_3 \amalg \ldots \amalg f_r$ is an embedding which is trivial in $BLM^m_{p_2,\ldots,p_r}$.*

Proof. Clearly, the defining representative of $e_*[g]$ in 1.3 restricts to the link homotopy trivial embedding $e_2 \amalg \ldots \amalg e_r$.

Conversely, let u have a representative f as above. After a suitable isotopy we may assume for every $2 \leq i \leq r$ that f_i agrees with the standard embedding e_i on $S^{p_i} - B_i$ and that every point in the remaining image $f_i(B_i)$ satisfies the inequalities $x_2 > \varepsilon$ and $x_3 > 1 - \varepsilon$; here B_i denotes a small ball in S^{p_i} around the point $(0, 1, 0, \ldots, 0)$. Then the image of $f_2 \amalg \ldots \amalg f_r$ intersects the halfspace $\mathbb{R}^m_\varepsilon$, defined by $x_2 > \varepsilon$, in a disjoint union of embedded balls whose complement we denote by C. Therefore we can compare the homology sequences of the pairs $(\vee B^{m-p_i}, \vee S^{m-p_i-1})$ and $(\mathbb{R}^m_\varepsilon, C)$ apply the Thom isomorphism and Whitehead theorems and deform the component map f_1 until it factors through the map e in 1.3. But now $f_1(S^p)$ cannot interfere anymore when we use a nulhomotopy of the remaining component maps to deform them within the positive x_3-halfspace until they agree completely with the standard embeddings e_i. ■

Now we are ready to draw conclusions from Theorem 2.13.

COROLLARY 2.15. *Assume $(*)$ $q \leq \frac{2}{3}m - 1$ and $(**)'$ $p \leq 3(m - q - 2) + 1$ and exclude the case $(m, p, q) = (2, 1, 0)$, $r \geq 3$ and the case when $m = 3$ and at least three p_j's equal 1.*

Then we have the exact sequence of abelian groups and group homomorphisms

$$\pi_p\left(\bigvee_{j=2}^{r} S^{m-p_j-1}\right) \xrightarrow{e_*} BLM^m_{(p)} \xrightarrow{B} BLM^m_{p_2,\ldots,p_r} \oplus \bigoplus_{j=2}^{r} B(m, p_1, p_j)$$

where $B[f_1 \amalg \ldots \amalg f_r] := ([f_2 \amalg \ldots \amalg f_r], \bigoplus_{j=2}^{r} \tilde{\beta}[f_1 \amalg f_j])$.

Proof. If $q \leq m - 3$, exactness follows from the last two results, and it remains only to show inductively that every class $[f]$ in the abelian semigroup $BLM^m_{(p)}$ has an additive inverse. For $r = 2$ this can be established as in [11], 1.19 (see also 1.11). If r is arbitrary, the class

$$[f] + \operatorname{incl}(-[f_2 \amalg \ldots \amalg f_r]) + \sum_{j=2}^{r} \operatorname{incl}(-[f_1 \amalg f_j])$$

lies in the image of e_* and hence has an inverse.

In the only remaining nontrivial case we have $m = 3$, $p = p_j = 1$ for some $j \geqslant 2$ and all other p_i's are zero; then e_* is even bijective and an inverse isomorphism is defined by the integer linking number $\alpha(f_1 \amalg f_j)$.

It follows from 1.6 and 3.12 that the claims of the corollary do indeed not hold in the two excluded cases.

§3. THE μ-INVARIANT

In this section we study some basic properties of the μ-invariant in the case of $r = 3$ components (see also [7]). Together with the exact sequence in [11] this suffices to complete the proof of our main theorem.

Assume $m \geq 3$ and put $\| p \| = p_1 + p_2 + p_3$ for any $(p) = (p_1, p_2, p_3)$. The definition of μ can be summarized by the commuting diagram

$$\begin{tikzcd}
\pi^S_{p-2m+3} & & \ker(\kappa_2) \arrow[ll, "\mu"'] \arrow[dll] \arrow[d] & \subset & BLM^m_{(p)} \arrow[d, "\wedge"] \\
\pi_p(S^{m-1} \vee S^{m-1}) \arrow[u, "\alpha"] \arrow[rr, hook, "\mathrm{incl}_*"] & & \pi_p(\tilde{C}_3(\mathbb{R}^m)) \arrow[rr, hook, "\mathrm{coll}^*"] \arrow[d, "\mathrm{proj}_*"] & & {[S^{p_1} \times S^{p_2} \times S^{p_3}, \tilde{C}_3(\mathbb{R}^m)]} \arrow[d, "\mathrm{proj}_*"] \\
& & \pi_p(\tilde{C}_2(\mathbb{R}^m)) \arrow[rr, hook, "\mathrm{coll}^*"] & & {[S^{p_1} \times S^{p_2} \times S^{p_3}, \tilde{C}_2(\mathbb{R}^m)].}
\end{tikzcd}$$

Here $\tilde{C}_n(\mathbb{R}^m)$ denotes the configuration space of ordered n-tuples of pairwise distinct points in $\mathbb{R}^m$. The projection $\mathrm{proj} : \tilde{C}_3(\mathbb{R}^m) \to \tilde{C}_2(\mathbb{R}^m)$ to the last two components is a locally trivial fibration with fiber $\mathbb{R}^m - \{*_1, *_2\} \sim S^{m-1} \vee S^{m-1}$ and with a section (cf. [1]). So the fiber inclusion induces an injective map between the indicated (standard) homotopy sets, and so does the map coll which collapses the subspace

$$S_2^{(p)} = \{(y_1, y_2, y_3) \in S^{p_1} \times S^{p_2} \times S^{p_3} \mid y_i = *_i \text{ for some } i\} \tag{3.2}$$

(see [15], Satz 12 and 20). Now given $[f] = [f_1 \amalg f_2 \amalg f_3]$ in $BLM^m_{(p)}$, consider the (standard) homotopy class $[\hat{f}\,]$ of the map

$$\hat{f} : S^{p_1} \times S^{p_2} \times S^{p_3} \longrightarrow \tilde{C}_3(\mathbb{R}^m)$$

defined by $\hat{f}(y_1, y_2, y_3) = (f_1(y_1), f_2(y_2), f_3(y_3))$. If $\hat{f} \mid S_2^{(p)}$ is nulhomotopic or, equivalently, if the desuspended α-invariants of the 2-component sublinkmaps of f vanish,

$$0 = \kappa_2[f] = \bigoplus_{1 \leq i \leq j \leq 3} \kappa_2[f_i \amalg f_j] \in \bigoplus_{1 \leq i \leq j \leq 3} \pi_{p_i + p_j}(S^{m-1}) \tag{3.3}$$

(compare [7], 2.7), then so do the values of $[\hat{f}\,]$ under $\mathrm{proj}_* = \mathrm{proj}_{2,3}$ and similarly under $\mathrm{proj}_{1,2}$ and $\mathrm{proj}_{1,3}$; thus there is a unique element $[\hat{\hat{f}}]$ in the kernel of

$$\mathrm{fold}_* : \pi_p(S^{m-1} \vee S^{m-1}) \longrightarrow \pi_{\|p\|}(S^{m-1}) \oplus \pi_{\|p\|}(S^{m-1}) \tag{3.4}$$

such that $\mathrm{coll}^* \circ \mathrm{incl}_* [\hat{\hat{f}}] = [\hat{f}]$. Now the Pontryagin-Thom procedure identifies $\pi_p(S^{m-1} \vee S^{m-1})$ (and $\ker(\mathrm{fold}_*)$, resp.) with the bordism group of framed links $M_1 \amalg M_2 \subset \mathbb{R}^p$ of codimension $m - 1$ (and such that $M_i \subset \mathbb{R}^{p_i}$ is nulbordant for

$i = 1, 2$, resp.). The resulting α-invariant (cf. [8], 1.3 or 1.7) fits into the commuting diagram

$$\begin{tikzcd}
 & \pi^S_{\|p\|-2m+3} & & \\
\pi_{\|p\|}(S^{2m-3}) \arrow[ur, "\pm E^r"] \arrow[dr, "{[\iota_1,\iota_2]}"'] & & & \\
 & \ker(\mathrm{fold}_*) \arrow[uu, "\alpha|"'] & \subset & \pi_{\|p\|}(S^{m-1} \vee S^{m-1}); \arrow[uull, "\alpha"']
\end{tikzcd} \tag{3.5}$$

(if $\|p\| \leq 3m - 6$ then all three arrows in the left hand triangle are isomorphisms).

We define the link homotopy invariant

$$\mu(f) = \alpha[\overset{*}{f}]. \tag{3.6}$$

If one of the component maps f_i of f happens to be constant, then clearly

$$\mu(f) = 0. \tag{3.7}$$

On the other hand, in order to get nontrivial examples—as well as a better understanding of what kind of geometric phenomena μ measures—assume that $f_2 \amalg f_3$ extends to a continuous map on unit balls

$$F = F_2 \amalg F_3 \, : \, B^{p_2+1} \amalg B^{p_3+1} \longrightarrow \mathbb{R}^m_+ \tag{3.8}$$

such that $F_2(B^{p_2+1}) \cap F_3(B^{p_3+1}) = \phi$. Then after small approximations the maps

$$f_1 \times F_j \, : \, S^{p_1} \times B^{p_j+1} \longrightarrow \mathbb{R}^m \times \mathbb{R}^m, \; j = 2, 3,$$

are transverse to the diagonal; the resulting inverse images $f_1 \pitchfork F_j$ are framed closed manifolds of dimensions $p_1 + p_j + 1 - m$ and give rise to a (generalized) 2-component link map into $\mathbb{R}^{p_1} \subset S^{p_1}$ with a well defined α-invariant

$$\alpha(f_1 \pitchfork F) \in \pi^S_{\|p\|-2m+3}. \tag{3.9}$$

This is constructed as in 3.5 above, but in an entirely different (dimension) setting.

Proposition 3.10. *If $p_2, p_3 \leq m - 3$, then for every $[f] = [f_1 \amalg f_2 \amalg f_3] \in \ker \kappa_2$ and for every extension F as in 3.8 we have*

$$\mu[f] = \alpha(f_1 \pitchfork F)$$

at least up to a fixed $\pm$ sign.

Proof. After small approximations we may assume that f_1 is smooth and transverse to F which in turn is a selftransverse immersion and, for every $y \in S^{p_j}$, $j = 2, 3$, restricts to an embedding of the line segment I_y which joins y to the base point $*$ in B^{p_j+1}. Indeed, since $p_j \leq m - 3$ and hence the double point manifold of F_j has a strictly smaller dimension than p_j, by Sard's theorem $*$ can be made to avoid the line through any two distinct points in B^{p_j+1} which have equal values under F_j.

Therefore, we can construct a continuously parametrized family ψ_y, $y \in S^{p_j}$, of isotopies of $\mathbb{R}^m$ with support near $F_j(B^{p_j+1})$ such that ψ_y maps y to $*_j = (j, 0, \ldots, 0)$ and most of the remaining interval I_y into the halfline

$$L_j = \{(j, x_2, 0, \ldots, 0) \in \mathbb{R}^m \,|\, x_2 < 0\}.$$

This induces a homotopy from $\hat{f}$ to the composed map

$$S^{p_1} \times S^{p_2} \times S^{p_3} \xrightarrow{h} \mathbb{R}^m - \{*_2, *_3\} \xrightarrow{\text{incl}} \tilde{C}_3(\mathbb{R}^m)$$

given by $h(y_1, y_2, y_3) = \psi_{y_2} \circ \psi_{y_3}(f_1(y_1))$. Clearly, this value lies in L_2 iff there is an element $\tilde{y}_2 \in I_{y_2} \subset \mathring{B}^{p_2+1}$ such that $(y_1, \tilde{y}_2) \in f_1 \pitchfork F_2$. Now $\tilde{y}_2$ determines y_2; also the first projection on $f_1 \pitchfork F_2 \subset S^{p_1} \times \mathring{B}^{p_2+1}$ is an immersion. So we see that h is transverse to L_2, and $h^{-1}(L_2) = N_2 \times S^{p_3}$ where the framed submanifold N_2 of $\mathbb{R}^{p_1} \times \mathbb{R}^{p_2} \subset S^{p_1} \times S^{p_2}$ is diffeomorphic to $f_1 \pitchfork F_2$. Since $\kappa_2[f]$ vanishes, N_2 allows a bordism which we use to modify $N_2 \times S^{p_3}$ near $y_3 = *$ until we get a framed closed manifold lying entirely in $\mathbb{R}^{P} = S^{p_1} \times S^{p_2} \times S^{p_3} - S_2^{(p)}$. This, together with a similar modification of $h^{-1}(L_3)$, corresponds to finding a homotopy factorization $\hat{\hat{f}}$ of h through coll as required in 3.6. Using the same bordisms in the intersection definition of α we see that the two modified submanifolds of $\mathbb{R}^{P}$ above (or, equivalently, $\hat{\hat{f}}$) have the same α-invariant as the singular link map $f_1 \pitchfork F$. ■

The requirement $p_2, p_3 \leq m - 3$ is unnecessary in the proof above at least if F can be chosen to be an embedding, e.g. in the setting where e_* is defined (see 1.3).

COROLLARY 3.11. *If $p_2, p_3 \leq m - 2$, then the diagram*

$$\begin{tikzcd} & \ker\left(\text{fold}_* : \pi_p\left(\bigvee_{j=2}^{3} S^{m-p_j-1}\right) \longrightarrow \bigoplus_{j=2}^{3} \pi_p(S^{m-p_j-1})\right) \arrow[ld, "\alpha|"'] \arrow[d, "e_*"] & \\ \pi^S_{|p|-2m+3} & \ker(\kappa_2) \arrow[l, "\mu"] & \subset \quad BLM^m_{(p)} \end{tikzcd}$$

commutes up to a fixed $\pm$ sign. (Here α is defined as in 3.5 or [8], 1.7).

This is an example of the "input-output analysis" proposed at the end of [7]. It has important consequences already in such fundamental cases as $p_1 = 1 \leq p_2 = p_3 = m - 2$. Indeed, the commutator of the standard generators of $\pi_1(S^1 \vee S^1)$ has a nontrivial α-invariant and hence a nontrivial value under the homomorphism e_* (cf. 1.4).

COROLLARY 3.12. *For $m \geq 3$ the semigroup $BLM^m_{1,m-2,m-2}$ is not commutative.*

Among the many compatibilities of μ (e.g. precomposing f with maps $g_i : (S^{n_i}, *) \to (S^{p_i}, *)$ corresponds to multiplying $\mu(f)$ with the suspensions $E^\infty[g_i]$ in π^S_*), additivity is presumably most basic (cf. 1.4).

LEMMA 3.13. *Let $m \geq 3$ and assume $p_1, p_2, p_3 \geq 1$ or $\leq m - 2$ or $p_2, p_3 \leq m - 3$. Then*

$$\mu : \ker \kappa_2 \longrightarrow \pi^S_{|p|-2m+3}$$

is a homomorphism of semigroups.

Proof. If $p_1, p_2, p_3 \geq 1$ and f, g are as in 1.5, we use affine contractions to deform the map $(\widehat{f+g})$ until it takes (y_1, y_2, y_3) to $(*_1, *_2, *_3)$ whenever at least two components y_i lie in the equator $S^{p_i}_+ \cap S^{p_i}_-$. If $\kappa_2(f)$ and $\kappa_2(g)$ vanish, there are similar nulhomotopies of $(\widehat{f+g})|(S^{p_i}_\pm \amalg S^{p_j}_\pm \amalg \text{equator})$ etc. which fit nicely together. The map at the end of the resulting homotopy takes (y_1, y_2, y_3) to the basepoint of $\tilde{C}_3(\mathbb{R}^m)$ whenever just *one* component y_i lies in the equator of its domain. In other words, $(\mathrm{coll}^*)^{-1}(\widehat{f+g})$ is the sum of $(\mathrm{coll}^*)^{-1}(\hat{f})$, $(\mathrm{coll}^*)^{-1}(\hat{g})$ and of six "mixed" terms coming from link maps where two component maps go into a different half space than the remaining one (which therefore may be contracted to a constant). Thus already $(\mathrm{coll}^*)^{-1} \circ \wedge\,|$ is additive on $\ker(\kappa_2)$.

The remaining cases are settled by 3.7 and Proposition 3.10. ■

If $BLM^m_{(p)}$ happens to be a full abelian group, then the splitting exact sequence of abelian groups

$$0 \longrightarrow \ker(\oplus\,\mathrm{rest}) \underset{\rho}{\overset{}{\rightleftarrows}} BLM^m_{(p)} \underset{\Sigma\,\mathrm{incl}}{\overset{\oplus\,\mathrm{rest}}{\rightleftarrows}} \bigoplus_{1 \le i < j \le 3} BLM^m_{p_i, p_j} \longrightarrow 0$$

yields the retraction $\rho = \mathrm{id} - (\Sigma\,\mathrm{incl}) \circ (\oplus\,\mathrm{rest})$ into $\ker(\oplus\,\mathrm{rest}) \subset \ker k_2$. We obtain the extended homomorphism

$$\mu : BLM^m_{(p)} \longrightarrow \pi^S_{|p| - 2m + 3}$$

which, for $p_2, p_3 \le m - 2$, commutes with $\pm e_*$ and the α-invariant defined on al of $\pi_p(\vee S^{m-p_j-1})$ by [8], 1.7 (compare the Corollary 3.11 above).

Finally let us return to the setting of the main theorem in the introduction and combine the discussion above with Corollary 2.15. We obtain the $\pm$ commuting diagram of abelian groups

$$\begin{tikzcd}
 & {} \arrow[d, dotted] \\
\pi_p\Big(\bigvee_{j=2}^{r} S^{m-p_j-1}\Big) \arrow[r, "\mathrm{fold} \oplus \bigoplus \alpha"] \arrow[d, "e_*"] & \bigoplus_{j=2}^{r} \pi_p(S^{m-p_j-1}) \oplus \bigoplus_{2 \le j < k \le r} \pi^S_{p+p_j+p_k-2m+3} \arrow[d, "\oplus e_* \oplus \mathrm{Id}"] \\
\widetilde{BLM}^m_{(p)} \arrow[r] \arrow[dr] & \bigoplus_{j=2}^{r} BLM^m_{p,p_j} \oplus \bigoplus_{2 \le j < k \le r} \pi^S_{p+p_j+p_k-2m+3} \arrow[d, "\oplus \tilde{\beta}"] \\
 & \bigoplus_{j=2}^{r} B(m, p, p_j)
\end{tikzcd}$$

At the right hand side we have basically a direct sum of exact sequences established in [11], Theorem 3.1. The upper horizontal arrow is the standard Hilton decomposition isomorphism. Thus the lower horizontal homomorphism (which restricts the map in our main theorem to the kernel $\widetilde{BLM}^m_{(p)}$ of rest: $BLM^m_{(p)} \to BLM^m_{(p_1, \ldots, p_r)}$) must also be bijective. Our main result follows now by induction over r.

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